Invertibility Preserving Linear Maps On Semi-Simple Banach Algebras

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Abstract

In this paper, we show that the essentiality of the scole of an ideal $\mathcal{B}$ of the semi-simple Banach algebra $\mathcal{A}$ implies that any invertibility preserving isomorphism $\phi : \mathcal{A} \to \mathcal{A}$ is a Jordan homomorphism. Specially if, the unitary semi-simple Banach algebra $\mathcal{A}$ has an essential minimal ideal then $\phi \mid_{soc(\mathcal{A})}$ is a Jordan homomorphism.

AMS Subject Classification(2000): 46J10, 46J15, 46J20

Key Words: Banach algebra, Jordan algebra, invertibility preserving, socle.

1 Introduction

Linear invertibility preserving maps of algebras, were noteworthy from years ago. For example, the famous theorem of Kahan-Zelasco which asserts that any invertibility preserving isomorphism into the scalar field is homomorphism. This problem discussed on different algebras previously. Let $\mathcal{A}$ be an unitary Banach algebra and $a \in \mathcal{A}$ is invertible. Then the inverse of a is denoted by a^{-1} and the set of all invertible elements of $\mathcal{A}$ is denoted by $Inv(\mathcal{A})$. Also, let $\mathcal{A}, \mathcal{B}$ are tow algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is a linear map. The ϕ is called an invertibility preserving map if,

$$a \in Inv(\mathcal{A}) \Rightarrow \phi(a) \in Inv(\mathcal{B}) \ \ forall \ \ a \in \mathcal{A}$$

The reader is referred to [1] for undefined terms and notations.

2 Main Results

The following lemma, is useful for the proof of our next theorem.
Lemma 2.1 [1] Let $\mathcal{A}, \mathcal{B}$ are unitary semi-simple Banach algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is an invertibility preserving isomorphism. Then,

$$\phi^{-1}(\phi(a^2) - \phi^2(a)).soc(\mathcal{A}) = 0 \quad forall \quad a \in \mathcal{A}$$

Moreover, if the $soc(\mathcal{A})$ is an essential ideal of $\mathcal{A}$, then ϕ is a Jordan homomorphism.(i.e. $\phi(a^2) = \phi^2(a) \;\; forall \;\; a \in \mathcal{A}$)
Now, let us to state of our main theorem:

Theorem 2.2 Let $\mathcal{A}$ is a semi-simple Banach algebra and $\phi : \mathcal{A} \to \mathcal{A}$ is an invertibility preserving isomorphism. Then ϕ is a Jordan homomorphism, whenever $\mathcal{A}$ has an ideal $\mathcal{B}$ that $soc(\mathcal{B})$ is an essential ideal.
proof. At first, we suppose that $\mathcal{A}$ is unitary. Since $soc(\mathcal{B})$ is an essential ideal of $\mathcal{A}$, $soc(\mathcal{A})$ is an essential ideal, too [1] and so ϕ is a Jordan homomorphism, by lemma 2.1. If now, $\mathcal{A}$ is not unitary $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ is an unitary semi-simple Banach algebra with $(a_1, \lambda_1).(a_2, \lambda_2) = (a_1 a_2 + \lambda_2 a_1 + \lambda_1 a_2, \lambda_1 \lambda_2)$. Let $\tilde{\phi}(a, \lambda) = (\phi(a), \lambda)$ for $(a, \lambda) \in \tilde{\mathcal{A}}$. The $\tilde{\phi} : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ is a well defined invertibility preserving isomorphism. If $soc(\mathcal{A}) = K$ and $(a, \lambda)soc(\tilde{\mathcal{A}}) = 0$, then $(a, \lambda).(k, 0) = 0$, for all $k \in K$ since $(soc(\mathcal{A}), 0) \subseteq soc(\mathcal{A}, 0) \subseteq soc(\tilde{\mathcal{A}})$. So for all $k \in K$, $ak = -\lambda k$ and therefore $\lambda = 0$. Because $\lambda \neq 0$ implies that $-\frac{a}{\lambda}k = k$, for all $k \in K$. So $-\frac{a}{\lambda}$ is a left unit of $\mathcal{A}$. Let d is an other left unit of $\mathcal{A}$. Then

$$(-\frac{a}{\lambda} - d)\mathcal{A} = 0 \Rightarrow (-\frac{a}{\lambda} - d)k = 0 \, for \, all \, k \in K \Rightarrow -\frac{a}{\lambda} = d$$

Note that K is an essential ideal. So $\mathcal{A}$ is unitary which contradicts our hypothesis. Therefore $a = 0$ and $soc(\tilde{\mathcal{A}})$ is an essential ideal. Now lemma 2.1 implies that,

$$\tilde{\phi}(a, \lambda)^2 = \tilde{\phi}^2(a, \lambda) \, for \, all \, (a, \lambda) \in \tilde{A}$$

But, for all $(a, \lambda) \in \tilde{A}$ we have,

$$\tilde{\phi}(a, \lambda)^2 = (\phi(a^2) + 2\lambda\phi(a), \lambda^2) \, and \, \tilde{\phi}^2(a, \lambda) = (\phi^2(a), \lambda)$$

Hence, for all $a \in \mathcal{A}$, $\phi^2(a) = \phi(a^2)$ and ϕ is a Jordan homomorphism.

Lemma 2.3 [4] Let $\mathcal{A}$ is an unitary semi-simple Banach algebra and $a \in \mathcal{A}$. Then
(i) $a \in soc(\mathcal{A})$ if and only if $\mid \sigma(xa) \mid \ < \infty \ forall \ x \in \mathcal{A}$
(ii) $a \in soc(\mathcal{A})$ if and only if there exists $n \in \mathbb{N}$ such that $\bigcap\limits_{t \in F} \sigma(x + ta) \subseteq \sigma(x)$ for all $x \in \mathcal{A}$ for which F is the set of n-tuples of $\mathbb{C} \setminus \{0\}$.

Lemma 2.4 Let $\phi : \mathcal{A} \to \mathcal{A}$ is a spectrum preserving isomorphism on the unitary semi-simple Banach algebra $\mathcal{A}$. Then $\phi(soc(\mathcal{A})) = soc(\mathcal{A})$.
proof. Let $a \in soc(\mathcal{A})$. Since ϕ is spectrum preserving, we have,

$$\sigma\left(\phi(y + ta)\right) = \sigma(y + ta) \ forall \ t \in \mathbb{C} \ and \ y \in \mathcal{A}$$

If now, $x = \phi(y)$ by the lemma 2.3, there exists $n \in \mathbb{N}$ such that,

$$\bigcap_{t \in F} \sigma(x + t\phi(a)) = \bigcap_{t \in F} \sigma\left(y + ta\right) \subseteq \sigma(x) \ forall \ x \in \mathcal{A}$$

where F is $n - tuples$ of $\mathbb{C} \backslash \{0\}$. Hence $\phi(a) \in soc(\mathcal{A})$ and so $\phi(soc(\mathcal{A})) \subseteq soc(\mathcal{A})$.
Now, we show that $soc(\mathcal{A}) \subseteq \phi(soc(\mathcal{A}))$. Let $a \in soc(\mathcal{A})$. Then there exists $b \in \mathcal{A}$ such that $\phi(b) = a$ and there exists $n \in \mathbb{N}$ such that,

$$\bigcap_{t \in F} \sigma(x + tb) = \bigcap_{t \in F} \sigma\left(\phi(x) + t\phi(b)\right) \subseteq \sigma(\phi(x)) = \sigma(x) \ forall \ x \in \mathcal{A}$$

where F is $n - tuples$ of $\mathbb{C} \backslash \{0\}$. This implies that $b \in soc(\mathcal{A})$.

Let us mention that if the Banach algebra $\mathcal{A}$ has an essential minimal ideal, then $soc(\mathcal{A})$ is essential.Thus we obtained the following consequence:

Corollary 2.5 If $\phi : \mathcal{A} \to \mathcal{A}$ is an invertibility preserving isomorphism on the unitary semi-simple Banach algebra $\mathcal{A}$ with an essential minimal ideal, then $\phi \mid_{soc(\mathcal{A})}$ is a Jordan homomorphism.

3 Acknowledgments

We wish to express our appreciation to our supervisor Dr. A. Assadi for his advice and encouragement in the preparation of this dissertation.

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